# The existence of best proximity point for a pair of multivalued mappings

R.C.Dimri,Pushpendra Semwal


## Absrtact

In this paper we establish a theorem on best proximity point for multivalued mappings satisfying the property-UC.Our result generalize and extend some of the results of Lin and Yang [5] and others.



**Key Words-**Best proximity point,Property-UC, MT-function,Cyclic map, Mulitivalued mapping.

## Mathematical Subject Classification-

47H10,46B20.


## 1. Introduction and Preliminaries

Let A and B be nonempty subsets of a metric space (X,d).Consider a mapping $T : A \cup B \rightarrow A \cup B$,T is called a cyclic map if T(A)$\subseteq B$ and T(B)$\subseteq A.x \in A$ is called a best proximity point of T in A if d(x,Tx) = d(A,B) is satisfied,where d(A,B) $=inf\{d(x,y) : x \in A, y \in B\}$.We denote by H the Hausdorff metrie H(A,B) = max{sup $d(a,B) : a \in A$, sup $d(A,b) : b \in B\}$ where d(a, B) $=inf\{d(a,b) : b \in B\}$. In 2005,Elderd et al.[1] proved the existence of a best proximity point for relatively nonexpansive mappings using the notion of proximal normal structure.In 2006,Eldred and Veeramani [2] proved the following existence theorem.

**Definition 1.1**. Let A and B be nonempty subsets of a metric space (X,d). The cyclic ( on A and B ) multivalued mapping T is said to be cyclic contraction if there exists a constant $k \in (0,1)$ such that

$$H(Tx,Ty) \leq kd(x,y) + (1-k)d(A,B)$$

for all $x \in A$ and $y \in B$

**Theorem1.2**[2]. Let A and B be nonempty closed convex subsets of a uniformaly convex Banach space.Suppose $f : A \cup B \rightarrow A \cup B$ is a cyclic contraction,that is,$f(A) \subseteq B$ and $f(B) \subseteq A$, and there exists $k \in (0,1)$ such that

$$d(fx,fy) \leq kd(x,y) + (1-k)d(A,B)$$

for every $x \in A, y \in B$
Then there exists a unique best proximity point in A. Further, for each $x \in A, \{f^{2n}x\}$ converges to the best proximity point.

**Definition1.3**[3-4] A function $\phi : [0,\infty) \rightarrow [0,1)$ is said to be an MT- function (or R-function) if $limsup_{s\rightarrow t^+}\phi(s) < 1$ for all $t \in [0,\infty)$.
It is obvious that if $\phi : [0,\infty) \rightarrow [0,1)$ is a nondecreasing function or a nonincreasing function, then $\phi$ is an MT-function. So the set of MT-functions is a quite rich class.
Very recently,Du [4] first proved some characterizations of MT-functions.

**Example1.4**[4] Let $\phi : [0.\infty) \rightarrow [0,1)$ be defined by

$$|x| = \begin{cases} \frac{sint}{t}; & \text{if } t \in (0,\frac{\pi}{2}] \\ 0; & otherwise. \end{cases}$$

since $lim_{s\rightarrow 0^+} sup\phi(s) = 1$, $\phi$ is not an MT-function.

**Theorem1.5**[4] Let $\phi : [0,\infty) \rightarrow [0,1)$ be a function.Then the following statements are equivalent.
(a) $\phi$ is an MT-function.
(b) For each $t \in [0,\infty)$,there exist $r_t^{(1)} \in [0,1)$

and $\epsilon_t^{(1)} > 0$ such that $\phi(s) \le r_t^{(1)}$ for all $s \in (t, t+\epsilon_t^{(1)})$.
(c) For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\epsilon_t^{(2)} > 0$ such that $\phi(s) \le r_t^{(2)}$, for all $s \in [t, t+\epsilon_t^{(2)}]$.
(d) For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\epsilon_t^{(3)} > 0$ such that $\phi(s) \le r_t^{(3)}$, for all $s \in [t, t+\epsilon_t^{(3)}]$.
(e) For each $t \in [0,\infty)$,there exist $r_t^{(4)} \in [0,1)$ and $\epsilon_t^{(4)} > 0$ such that $\phi(s) \le r_t^{(4)}$ for all $s \in (t, t+\epsilon_t^{(4)})$.
(f) For any nonincreasing sequence $\{x_n\}_{n\in N}$ in $[0,\infty)$,we have $0 \le sup_{n\in N}\phi(x_n) \le 1$.

(g) $\phi$ is a function of contractive factor ; that is , for any strictly decreasing sequence $\{x_n\}_{n\in N}$ in$[0,\infty)$,we have $0 \le sup_{n\in N}\phi(x_n) \le 1$.

**Definition1.6**[6] Let A and B be nonempty subsets of a metric space (X,d).Then (A,B) is said to satisfy the property UC if the following holds:
If $\{x_n\}$ and $\{x_n^{'}\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $lim_{n\to\infty}d(x_n, y_n) = d(A,B)$ and $lim_{n\to\infty}d(x_n^{'}, y_n) = d(A,B)$, then $lim_{n\to\infty}d(x_n, x_n^{'}) = d(A,B)$.

**Lemma1.7**[5] Let A and B be nonempty subsets of a metric space (X,d) with property UC and let $\{x_n\}$ be a sequence in A.If there exists a sequence $\{y_n\}$ in B such that $lim_{n\to\infty}d(x_n, y_n) = d(A,B)$ and $lim_{n\to\infty}d(x_{n+1}, y_n) = d(A,B)$, then $\{x_n\}$ is a Cauchy sequence.

## 2. Main Result

**Theorem2.1** Let A and B be a nonempty subsets of a metric space (X,d) such that (A,B) satisfies the property UC and A is complete.Let F and G are two (on A and B) multivalued mappings such that Fx$\subseteq B$, for all $x \in A$ and Gy$\subseteq A$, for all $y \in B$.If there exists a nondecreasing function $\mu : [0,\infty) \to [0,1)$ and an MT-function $\phi : [0,\infty) \to [0,1)$ such that

$$H(Fx, Gy) \le \frac{1}{2}\phi(\mu(d(x,y)))[d(x,Fx) + d(y,Gy)]+ [1-\phi(\mu(d(x,y)))]d(A,B) \quad (1)$$

for all $x \in A$ and $y \in B$, then F and G has comman best proximity point in A.

**Proof.** Fix $x_0 \in A$.Let $x_1 \in Fx_0 \subseteq B$.There exists $x_2 \in Gx_1 \subseteq A$ such that

$$\begin{aligned} d(x_1,x_2) &\le d(x_1, Gx_1) + k \\ &\le h(Fx_0, Gx_1) + k \\ &\le H(Fx_0, Gx_1) + k \\ &\le \frac{1}{2}\phi(\mu(d(x_0,x_1)))[d(x_0,Fx_0) + d(x_1,Gx_1)]+ \\ &[1-\phi(\mu(d(x_0,x_1)))]d(A,B) + k \\ &\le \frac{1}{2}\phi(\mu(d(x_0,x_1)))[d(x_0,x_1) + d(x_1,x_2)]+ \\ &[1-\phi(\mu(d(x_0,x_1)))]d(A,B) + k \end{aligned}$$

which implies

$$[1-\frac{1}{2}\phi(\mu(d(x_0,x_1)))]d(x_1,x_2) \le [1-\frac{1}{2}\phi(\mu(d(x_0,x_1)))]$$

$$d(x_0,x_1) + [1-\phi(\mu(d(x_0,x_1)))]d(A,B) + k \quad (2)$$

From (2), we obtain

$$\begin{aligned} d(x_1,x_2) \le \frac{\phi(\mu(d(x_0,x_1)))}{2-\phi(\mu(d(x_0,x_1)))}d(x_0,x_1)+ \\ [1-\frac{\phi(\mu(d(x_0,x_1)))}{2-\phi(\mu(d(x_0,x_1)))}]d(A,B) + k \end{aligned}$$

From (1) again, we have

$$\begin{aligned} d(x_2,x_3) &= H(Gx_1, Fx_2) + k \\ &= H(Fx_2, Gx_1) + k \\ &\le \frac{1}{2}\phi(\mu(d(x_2,x_1)))[d(x_2,Fx_2)+ \\ &d(x_1,Gx_1)] + [1-\phi(\mu(d(x_2,x_1)))] \\ &d(A,B) + k \\ &= \frac{1}{2}\phi(\mu(d(x_2,x_1)))[d(x_2,x_3) + d(x_1,x_2)] \\ &+ [1-\phi(\mu(d(x_2,x_1)))]d(A,B) + k \end{aligned}$$

which implies

$$[1-\frac{1}{2}\phi(\mu(d(x_2,x_1)))]d(x_2,x_3) \leq [1-\frac{1}{2}\phi(\mu(d(x_2,x_1)))] d(x_1,x_2)+ [1-\phi(\mu(d(x_2,x_1)))] d(A,B)+k$$

and hence

$$d(x_2,x_3) \leq \frac{\phi(\mu(d(x_2,x_1)))}{2-\phi(\mu(d(x_2,x_1)))}d(x_2,x_1)+ [1-\frac{\phi(\mu(d(x_2,x_1)))}{2-\phi(\mu(d(x_2,x_1)))}]d(A,B)+k$$

By induction, we have

$$d(x_n,x_{n+1}) \leq \frac{\phi(\mu(d(x_{n-1},x_n)))}{2-\phi(\mu(d(x_{n-1},x_n)))}d(x_{n-1},x_n) +[1-\frac{\phi(\mu(d(x_{n-1},x_n)))}{2-\phi(\mu(d(x_{n-1},x_n)))}]d(A,B)+k \quad (3)$$

Since $\phi$ is an MT-function, we obtain

$$0 \leq sup\phi(\mu(d(x_n,x_{n+1}))) < 1$$

Let $\alpha = sup\phi(\mu(d(x_n,x_{n+1})))$. So 0$\leq \alpha < 1$.Since

$$\phi(\mu(d(x_n,x_{n+1}))) \leq \alpha \quad (4)$$

Thus,

$$2-\phi(\mu(d(x_n,x_{n+1}))) \geq 2-\alpha, for\ all\ n \in N \quad (5)$$

Therefore, by (4) and (5), we get

$$\frac{\phi(\mu(d(x_{n-1},x_n)))}{2-\phi(\mu(d(x_{n-1},x_n)))} \leq \frac{\alpha}{2-\alpha} \quad (6)$$

From (6),

$$0 \leq sup\frac{\phi(\mu(d(x_{n-1},x_n)))}{2-\phi(\mu(d(x_{n-1},x_n)))} \leq \frac{\alpha}{2-\alpha} < 1$$

Let

$$\beta = sup\frac{\phi(\mu(d(x_{n-1},x_n)))}{2-\phi(\mu(d(x_{n-1},x_n)))}$$

Then $\beta \in [0,1)$ It follows from (3) that

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \beta d(x_{n-1},x_n)+(1-\beta)d(A,B)+k \\ &\leq \beta^2 d(x_{n-2},x_{n-1})+(1-\beta^2)d(A,B)+k^2 \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \beta^n d(x_0,x_1)+(1-\beta^n)d(A,B)+k^n \end{aligned}$$

Since $\beta \in [0,1)$, we have $lim_{n\to\infty}\beta^n = 0$. So the last inequality implies

$$lim_{n\to\infty}d(x_n,x_{n+1}) = d(A,B)$$

This implies

$$lim_{n\to\infty}d(x_{2n},x_{2n+1}) = d(A,B),$$

$$lim_{n\to\infty}d(x_{2n+2},x_{2n+1}) = d(A,B)$$

Since $x_{2n} \in Gx_{2n-1} \subseteq A, x_{2n+2} \in Gx_{2n+1} \subseteq A$ and $x_{2n+1} \in Fx_{2n} \subseteq B$, by Lemma1.7,$x_{2n}$ is a Cauchy sequence.By completeness of A,there exists $z \in A$ such that $lim_{n\to\infty}d(z,x_{2n}) = 0$. Since d(A,B)$\leq d(z,x_{2n+1}) \leq d(z,x_{2n})+d(x_{2n},x_{2n+1}) \leq d(A,B)$,we have $lim_{n\to\infty}d(z,Fx_{2n}) = d(A,B)$
Since d(A,B)$\leq d(x_{2n},Gz) \leq d((A,B)$, we have $lim_{n\to\infty}d(z,Gz) = d(A,B)$.
Hence d(z,Fz) = d(z,Gz) = d(A,B)

## References


[1] A.A.Eldred,W.A.Kirk,and P.Veeramani,"Proximal normal structure and relatively nonexpansive mappings,"Studia Mathematica,vol.171,no.3,pp.283-293, 2005.

[2]A.A.Eldred and P.Veeramani,"Existence and convergence of best proximity points,"Journal of Mathematical Analysis and Applications,vol.323,no.2,pp. 1001-1006,2006.

[3]W.-S. Du,"Some new results and generalisations in metric fixed point theory," Nonlinear Analysis, vol.73,pp.1439-1466,2010.

[4]W.-S. Du,"On coincidence point and fixed point theorems for nonlinear multivalued maps," Topology and its Applications, vol.159,pp.49-56,2012.

[5]I-J Lin and S-H Yang,"Some existence and convergence theorem of best proximity point for new nonlinear cyclic maps" Int. Journal of Math.Analysis, vol 6,no 52,pp.2563-2578,2012

[6]K.Neammanee and A.Kaewkhao,"Fixed points and best proximity points for multivalued mapping satisfying cyclical condition," Int.J.of Mathematical sciences and Applications,vol.1,pp.1-9,2011.

[7]T. Suzuki,M.Kikkawa and C.Vetro,"The existence of best proximity points in metric spaces with the property UC,"Nonlinear Analysis,vol.71,pp. 2918-2926,2009.

Email*. dimrirc@gmail.com
Email**. psrsdm@gmail.com
Department of Mathematics, H.N.B.Garhwal University, Srinagar Garhwal